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COVARIANCE LSM-ANALYSIS OF BIPERIODIC NONSTATIONARY VIBRATION SIGNALS

I. M. Javorskyj [1, 2], O. Y. Dzeryn [1], R. M. Yuzefovych [1, 3]

[1] **Karpenko Physico-Mechanical Institute of the NAS of Ukraine, Lviv;**

[2] **Telecommunication Institute of University of Technology and Life Science, Bydgoszcz, Poland;**

[3] **Lviv Polytechnic National University**

E-mail: dzeryn_oksana@ukr.net

The estimators of covariance function of biperiodically correlated random processes – mathematical models of vibration signals with binary stochastic recurrence, obtained with using the least squares method (LSM), are analyzed. It was shown that these estimators are unbiased and consistent under the condition of correlation relationships decaying with the bias rise. The main LSM-estimator advantage over the component estimator is the absence of leakage effects, which can cause significant errors of covariance characteristics estimation when combination frequencies have close values. Formulae obtained in this paper for statistic characteristics of LSM-estimator give an opportunity to calculate processing errors for specific signal types and also compare them with the errors of component estimation.

Key words: *biperiodic nonstationary vibration signal, covariance function estimator, least squares method, asymptotical unbiasedness and consistency, leakage.*

КОРЕЛЯЦІЙНИЙ МНК-АНАЛІЗ БІПЕРІОДИЧНО НЕСТАЦІОНАРНИХ ВІБРАЦІЙНИХ СИГНАЛІВ

І. М. Яворський [1, 2], О. Ю. Дзерин [1], Р. М. Юзефович [1, 3]

[1] **Фізико-механічний інститут ім. Г. В. Карпенка НАН України, Львів;**

[2] **Інститут телекомунікації Технологічно-природничого університету, Бидгощ, Польща;**

[3] **Національний університет "Львівська політехніка"**

Проаналізовано оцінки кореляційної функції біперіодично корельованих випадкових процесів – математичних моделей вібраційних сигналів з подвійною стохастичною повторюваністю, які знаходять методом найменших квадратів. Доведено, що такі оцінки є асимптиотично незміщеними й слушними за умови заникання кореляційних зв'язків з ростом зсуву. Показано, що цим методом можна уникнути систематичних похибок оцінювання, пов'язаних з ефектом просочування.

Ключові слова: *біперіодично корельовано випадкові процеси, оцінка кореляційної функції, метод найменших квадратів, асимптотична незміщеність і слушність, просочування.*

Shafts misalignment, imbalance, inner or outer rolling bearing ring skews are developing distributed faults can cause appearance of cavings, delaminations, cracks on its inner or outer tracks – so called local faults. In this case vibration signals have the properties of biperiodical nonstationarity [1, 2], that is, their mean function $m(t)=E\xi(t)$, E – mean operator, and covariance function $b(t,u)=E\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t+u)$, $\overset{\circ}{\xi}(t)=\xi(t)-m(t)$, can be represented by Fourier series:

$$m(t)=\sum_{k.l\in Z} m_{kl}e^{i\lambda_{kl}t}\ , \tag{1}$$

$$b(t,u)=\sum_{k,l\in Z} B_{kl}(u)\, e^{i\lambda_{kl}t}\ , \tag{2}$$

where $\lambda_{kl}=k\frac{2\pi}{T_1}+l\frac{2\pi}{T_2}$, T_1 and T_2 are the main periods determined respectively by shaft speed and speed of bodies of rotation. Vibrations caused by the tooth gear have similar properties. Then quantity T_1 is determined in advance by the shaft speed, and second T_2 is determined by the meshing speed.

It follows from formulae (1) and (2) that in the determined and stochastic parts of biperiodically correlated random processes model (BPCRP) the properties of oscillations of the main frequencies $\lambda_{10}=\frac{2\pi}{T_1}$, $\lambda_{01}=\frac{2\pi}{T_2}$ and multiple to them are described. Also combination frequencies λ_{kl}, caused by the interaction of both rotation moves are described. Values of λ_{kl} can be very close to each other. That's why estimating the mean and covariance function by component method [1–4] can cause significant leakage effects. Possibility to avoid these errors by using the least square method (LSM) is shown in [2]. LSM-estimators of mean are unbiased and theirs variances on the long length of realization $\theta(\theta\gg M_1T_1,\ \ \theta\gg M_2T_2, M_1, M_2\in N)$ differ insignificantly from variances of component estimators. This paper is devoted to the analysis of LSM-estimators properties of BPCRP covariance function.

Let us find covariance function estimator minimizing functional

$$F\left[\hat{B}_{00},\hat{B}^{c}_{01}(u),...,\hat{B}^{c}_{0N_1}(u),...,\hat{B}^{s}_{kl}(u),...,\hat{B}_{N_1N_1}(u)\right]=$$
$$=\int_0^{\theta}\left[\zeta(t,u)-\sum_{l,k=-N_2}^{N_2}\hat{B}_{kl}(u)e^{i\lambda_{kl}t}\right]^2 dt\ , \tag{3}$$

where

$$\hat{B}_{kl}(u)=\frac{1}{2}\left[\hat{B}^{c}_{kl}(u)-i\hat{B}^{s}_{kl}(u)\right],\quad \zeta(t,u)=\left[\xi(t)-\hat{m}(t)\right]\left[\xi(t+u)-\hat{m}(t+u)\right],$$

and θ is the realization length. Functional (3) contains unknown functions $\hat{B}_{kl}(u)$ that will be found considering necessary conditions of statistic (3) minimum existence. Let us rewrite covariance function statistic in a real form. Taking into account $B_{-k,-l}(u)=$ $=\overline{B_{kl}(u)}$, where "$\bar{\ }$" is the conjugation sign we obtain for covariance function

$$b(t,u)=B_{00}(u)+\sum_{k=0}^{N_2}\sum_{l=1}^{N_2}\left[B^{c}_{kl}(u)\cos\lambda_{kl}t+B^{s}_{kl}(u)\sin\lambda_{kl}t\right]+$$
$$+\sum_{l=0}^{N_2}\sum_{k=1}^{N_2}\left[B^{c}_{k,-l}(u)\cos\lambda_{k,-l}t+B^{s}_{k,-l}(u)\sin\lambda_{k,-l}t\right]. \tag{4}$$

Covariance function of BPCRP as is evident contains additional parts with frequencies $k\frac{2\pi}{T_1}$ and $l\frac{2\pi}{T_2}$, $k,l\in\left[1, N_2\right]$, and combination harmonics with frequencies

$k\frac{2\pi}{T_1} \pm l\frac{2\pi}{T_2}$. Let us express estimator $\hat{b}(t,u)$ in the form of series similarly to (4) and rewrite functional (3) in the form

$$F\left[\hat{B}_{00}(u), \hat{B}^c_{kl}(u), \hat{B}^s_{kl}(u), \hat{B}^c_{k,-l}(u), \hat{B}^s_{k,-l}(u)\right] = \int_0^\theta \Bigg[\zeta(t,u) - \Bigg[\hat{B}_{00}(u) + \sum_{k=0}^{N_2} \sum_{l=1}^{N_2} \Big[\hat{B}^c_{kl}(u) \times$$

$$\times \cos\lambda_{kl}t + \hat{B}^s_{kl}(u)\sin\lambda_{kl}t \Big] + \sum_{l=0}^{N_2} \sum_{k=1}^{N_2} \Big[\hat{B}^c_{k,-l}(u)\cos\lambda_{k,-l}t + \hat{B}^s_{k,-l}(u)\sin\lambda_{k,-l}t \Big] \Bigg] \Bigg]^2 dt \,. \quad (5)$$

These equations are necessary conditions of functional (5) minimum existence:

$$\frac{\partial F(u)}{\partial \hat{B}_{00}(u)} = 0 \,, \quad \left. \begin{array}{l} \dfrac{\partial F(u)}{\partial \hat{B}^c_{kl}(u)} = 0, \\ \dfrac{\partial F(u)}{\partial \hat{B}^s_{kl}(u)} = 0 \end{array} \right\} \begin{array}{l} k = \overline{0, N_1} \\ l = \overline{1, N_1} \end{array} \quad \left. \begin{array}{l} \dfrac{\partial F(u)}{\partial \hat{B}^c_{k,-l}(u)} = 0, \\ \dfrac{\partial F(u)}{\partial \hat{B}^s_{k,-l}(u)} = 0 \end{array} \right\} \begin{array}{l} k = \overline{1, N_1} \\ l = \overline{0, N_1} \end{array} .$$

In expanded form these equations are written as:

$$\theta\hat{B}_{00}(u) + \sum_{k=0}^{N_2}\sum_{l=1}^{N_2}\left[\hat{B}^c_{kl}(u)\int_0^\theta \cos\lambda_{kl}tdt + \hat{B}^s_{kl}(u)\int_0^\theta \sin\lambda_{kl}tdt\right] +$$

$$+\sum_{l=0}^{N_2}\sum_{k=1}^{N_2}\left[\hat{B}^c_{k,-l}(u)\int_0^\theta \cos\lambda_{k,-l}tdt + \hat{B}^s_{k,-l}(u)\int_0^\theta \sin\lambda_{k,-l}tdt\right] = \int_0^\theta \zeta(t,u)\,dt \,,$$

$$\hat{B}_{00}(u)\int_0^\theta \cos\lambda_{rp}tdt + \sum_{k=0}^{N_2}\sum_{l=1}^{N_2}\left[\hat{B}^c_{kl}(u)\int_0^\theta \cos\lambda_{kl}t\cos\lambda_{rp}tdt + \hat{B}^s_{kl}(u)\int_0^\theta \sin\lambda_{kl}t\cos\lambda_{rp}tdt\right] +$$

$$+\sum_{l=0}^{N_2}\sum_{k=1}^{N_2}\left[\begin{array}{l}\hat{B}^c_{k,-l}(u)\int_0^\theta \cos\lambda_{k,-l}t\cos\lambda_{rp}tdt + \\ +\hat{B}^s_{k,-l}(u)\int_0^\theta \sin\lambda_{k,-l}t\cos\lambda_{rp}tdt\end{array}\right] = \int_0^\theta \zeta(t,u)\cos\lambda_{rp}tdt \,,$$

$$\hat{B}_{00}(u)\int_0^\theta \sin\lambda_{rp}tdt + \sum_{k=0}^{N_2}\sum_{l=1}^{N_2}\left[\hat{B}^c_{kl}(u)\int_0^\theta \cos\lambda_{kl}t\sin\lambda_{rp}tdt + \hat{B}^s_{kl}(u)\int_0^\theta \sin\lambda_{kl}t\sin\lambda_{rp}tdt\right] +$$

$$+\sum_{l=0}^{N_2}\sum_{k=1}^{N_2}\left[\hat{B}^c_{k,-l}(u)\int_0^\theta \cos\lambda_{k,-l}t\sin\lambda_{rp}tdt + \hat{B}^s_{k,-l}(u)\int_0^\theta \sin\lambda_{k,-l}t\sin\lambda_{rp}tdt\right] =$$

$$= \int_0^\theta \zeta(t,u)\sin\lambda_{rp}tdt \,.$$

In these equations $p = \overline{-N_2, N_2}$, $p \neq 0$. To simplify the analysis of the equation system solutions we reassign frequencies according to Tables 1 and 2.

Table 1. Frequencies of harmonical part of estimator $\hat{m}(t)$

$\Lambda_{N_2,-N_2}$	$\Lambda_{N_2,-N_2+1}$	…	$\Lambda_{N_2,-1}$	$\Lambda_{N_2 0}$	$\Lambda_{N_2 1}$	$\Lambda_{N_2 2}$	…	$\Lambda_{N_2 N_2}$
$\Lambda_{N_2-1,-N_2}$	$\Lambda_{N_2-1,-N_2+1}$	…	$\Lambda_{N_2-1,-1}$	$\Lambda_{N_2-1,0}$	$\Lambda_{N_2-1,1}$	$\Lambda_{N_2-1,2}$	…	Λ_{N_2-1,N_2}
…	…	…	…	…	…	…	…	…
$\Lambda_{2,-N_2}$	$\Lambda_{2,-N_2+1}$	…	$\Lambda_{2,-1}$	$\Lambda_{2,0}$	$\Lambda_{2,1}$	$\Lambda_{2,2}$	…	Λ_{2,N_2}
$\Lambda_{1,-N_2}$	$\Lambda_{1,-N_2+1}$	…	$\Lambda_{1,-1}$	$\Lambda_{1,0}$	$\Lambda_{1,1}$	$\Lambda_{1,2}$	…	Λ_{1,N_2}
					$\Lambda_{0,1}$	$\Lambda_{0,2}$	…	Λ_{0,N_2}

Table 2. Reassigned frequencies of estimator $\hat{m}(t)$

$\omega_{2N_2(N_2+1)}$	…	$\omega_{N_2(2N_2+1)}$	$\omega_{N_2(2N_2+1)}$	$\omega_{N_2^2+1}$	$\omega_{N_2^2+2}$	…	$\omega_{N_2^2+N_2-1}$	$\omega_{N_2(N_2+1)}$
$\omega_{N_2(2N_2+1)-1}$	…	$\omega_{2N_2^2}$	$\omega_{2N_2^2-1}$	$\omega_{(N_2-1)N_2+1}$	$\omega_{(N_2-1)N_2+2}$	…	$\omega_{N_2^2-1}$	$\omega_{N_2^2}$
	…		…	…	…	...	…	…
$\omega_{N_2(N_2+3)+2}$	…	$\omega_{N_2(N_2+2)+2}$	$\omega_{N_2(N_2+2)+2}$	ω_{2N_2+1}	ω_{2N_2+2}	…	ω_{3N_2-1}	ω_{3N_2}
$\omega_{N_2(N_2+2)+1}$	…	$\omega_{N_2(N_2+1)+2}$	$\omega_{N_2(N_2+1)+1}$	ω_{N_2+1}	ω_{N_2+2}	…	ω_{2N_2-1}	ω_{2N_2}
				ω_1	ω_2	…	ω_{N_2-1}	ω_{N_2}

Functional (5) after these reassignments has the form

$$F\left[\hat{B}_{00},\hat{B}^c_1(u),...,\hat{B}^c_L(u),\hat{B}^s_1(u),...,\hat{B}_L(u)\right]=$$

$$=\int_0^\theta\left[\zeta(t,u)-\left[\hat{B}_{00}(u)+\sum_{r=1}^{L}\left[\hat{B}^c_2(u)\cos\omega_2 t+\hat{B}^c_2(u)\sin\omega_2 t\right]\right]\right]^2 dt\,,$$

where $L=2N_2(N_2+1)$. Necessary conditions of its minimum existence:

$$\frac{\partial F}{\partial\hat{B}_0(u)}=0\,,\quad \frac{\partial F}{\partial\hat{B}^c_r(u)}=0\,,\quad \frac{\partial F}{\partial\hat{B}^s_r(u)}=0\,,\quad r=\overline{1,L}\,,$$

now in expanded form are the following:

$$\theta\hat{B}_0(u)+\sum_{k=1}^{L}\left[\hat{B}^c_k(u)\int_0^\theta\cos\omega_k t dt+\hat{B}^s_k(u)\int_0^\theta\sin\omega_k t dt\right]=\int_0^\theta\zeta(t,u)\,dt\,,$$

$$\hat{B}_0(u)\int_0^\theta\cos\omega_r t dt+\sum_{k=1}^{L}\left[\begin{array}{l}\hat{B}^c_k(u)\int_0^\theta\cos\omega_k t\cos\omega_r t dt+\\+\hat{B}^s_k(u)\int_0^\theta\sin\omega_k t\cos\omega_r t dt\end{array}\right]=\int_0^\theta\zeta(t,u)\cos\omega_r t dt\,,$$

$$\hat{B}_0(u)\int_0^\theta\sin\omega_r t dt+\sum_{k=1}^{L}\left[\begin{array}{l}\hat{B}^c_k(u)\int_0^\theta\cos\omega_k t\sin\omega_r t dt+\\+\hat{B}^s_k(u)\int_0^\theta\sin\omega_k t\sin\omega_r t dt\end{array}\right]=\int_0^\theta\zeta(t,u)\sin\omega_r t dt\,. \quad (6)$$

Introducing reassignments

$$c_{rk} = \frac{1}{\theta}\int_0^{\theta} \cos\omega_r t \cos\omega_k t dt \,,\quad s_{rk} = \frac{1}{\theta}\int_0^{\theta} \sin\omega_r t \sin\omega_k t dt \,,\quad a_{rk} = \frac{1}{\theta}\int_0^{\theta} \cos\omega_r t \sin\omega_k t dt \,,$$

and also

$$\tilde{B}_0(u) = \frac{1}{\theta}\int_0^{\theta} \zeta(t)\,dt \,,\quad \tilde{B}_r^c(u) = \frac{1}{\theta}\int_0^{\theta} \zeta(t,u)\cos\omega_r t dt \,,\quad \tilde{B}_r^s(u) = \frac{1}{\theta}\int_0^{\theta} \zeta(t,u)\sin\omega_r t dt \,,$$

rewrite system (6) in the matrix form:

$$D\hat{B}(u) = \tilde{B}(u) \,, \tag{7}$$

where

$$\hat{B}(u) = \left[\hat{B}(u)_0, \hat{B}(u)_1^c, ..., \hat{B}(u)_L^c, \hat{B}(u)_1^s, ..., \hat{B}(u)_L^s\right]^T ,$$

$$\tilde{B}(u) = \left[\tilde{B}(u)_0, \tilde{B}(u)_1^c, ..., \tilde{B}(u)_L^c, \tilde{B}(u)_1^s, ..., \tilde{B}(u)_L^s\right]^T ,$$

$$D = \begin{bmatrix} 1 & c_{01} & c_{02} & \cdots & c_{0L} & a_{01} & a_{02} & \cdots & a_{0L} \\ c_{10} & c_{11} & c_{12} & \cdots & c_{1L} & a_{11} & a_{12} & \cdots & a_{1L} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{L0} & c_{L1} & c_{L2} & \cdots & c_{LL} & a_{L1} & a_{L2} & \cdots & a_{LL} \\ a_{01} & a_{11} & a_{21} & \cdots & a_{L1} & s_{11} & s_{12} & \cdots & s_{1L} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{0L} & a_{1L} & a_{2L} & \cdots & a_{LL} & s_{L1} & s_{L2} & \cdots & s_{LL} \end{bmatrix} = \left[d_{rk}\right], \quad \begin{matrix} r = \overline{1, 2L+1}, \\ k = \overline{1, 2L+1}. \end{matrix} \tag{8}$$

Solution of system (7) can be written as:

$$\hat{B}(u) = D^{-1}\tilde{B}(u) = \frac{[A_{rk}]^T}{|D|}\tilde{B}(u) \,,$$

where $|D|$ is the matrix (8) determinant and $[A_{rk}]^T$ is the transposed matrix of algebraic complements. It follows from the above:

$$E\hat{B}(u) = \frac{[A_{rk}]^T}{|D|} E\tilde{B}(u) \,.$$

Matrix elements $E\tilde{B}(u)$ are determined by the formulae

$$E\tilde{B}_0(u) = \frac{1}{\theta}\int_0^{\theta} m_{\zeta}(t,u)\,dt \,,\quad E\tilde{B}_r^c(u) = \frac{1}{\theta}\int_0^{\theta} m_{\zeta}(t,u)\cos\omega_r t dt \,,$$

$$E\tilde{B}_r^s(u) = \frac{1}{\theta}\int_0^{\theta} m_{\zeta}(t,u)\sin\omega_r t dt \,, \tag{9}$$

where

$$m_{\zeta}(t,u) = E\left[\xi(t) - \hat{m}(t)\right]\left[\xi(t+u) - \hat{m}(t+u)\right] = b(t,u) - \varepsilon_{\zeta}(t,u) \,,$$

herewith

$$\varepsilon_{\zeta}(t,u) = E\left[\overset{\circ}{\hat{m}}(t)\overset{\circ}{\xi}(t+u) + \overset{\circ}{\hat{m}}(t)\overset{\circ}{\hat{m}}(t+u) - \overset{\circ}{\hat{m}}(t)\overset{\circ}{\xi}(t)\right], \tag{10}$$

and estimators $\overset{\circ}{\hat{m}}(t)$ are determined by the ratios:

$$\overset{\circ}{\hat{m}}(t)=\hat{m}(t)-m(t)=\frac{1}{|M|}\left[\left[\frac{1}{\theta}\int_0^\theta \overset{\circ}{\xi}(s)ds\right]f_0(t)+\sum_{p=1}^{\tilde{L}}\left[\left[\frac{1}{\theta}\int_0^\theta \overset{\circ}{\xi}(s)\cos\omega_p sds\right]\tilde{f}_p(t)+\right.\right.$$

$$\left.\left.+\left[\frac{1}{\theta}\int_0^\theta \overset{\circ}{\xi}(s)\sin\omega_p sds\right]\tilde{f}_{p+\tilde{L}}(t)\right]\right], \qquad (11)$$

where

$$f_p(t)=M_{p+1,1}+\sum_{q=1}^{\tilde{L}}\left[M_{p+1,q+1}\cos\omega_q t+M_{p+1,q+L+1}\sin\omega_q t\right]$$

and M_{pq} are algebraic complements of the equation system matrix $M=[m_{ik}]$ of order $(2\tilde{L}+1)\times(2\tilde{L}+1)$, solutions of this matrix are Fourier coefficients of mean [2].

Taking into account series

$$b(t,u)=B_0(u)+\sum_{k=1}^{L}\left[B_k^c(u)\cos\omega_k t+B_k^s(u)\sin\omega_k t\right],$$

we obtain

$$\frac{1}{\theta}\int_0^\theta b(t,u)dt=B_0(u)d_{11}+\sum_{k=1}^{L}\left[B_k^c(u)d_{1,k+1}+B_k^s(u)d_{1,L+k+1}\right],$$

$$\frac{1}{\theta}\int_0^\theta b(t,u)\cos\omega_r tdt=B_0(u)d_{r+1,1}+\sum_{k=1}^{L}\left[B_k^c(u)d_{r+1,k+1}+B_k^s(u)d_{r+1,L+k+1}\right],$$

$$\frac{1}{\theta}\int_0^\theta b(t,u)\sin\omega_r tdt=B_0(u)d_{L+r+1,1}+\sum_{k=1}^{L}\left[B_k^c(u)d_{L+r+1,k+1}+B_k^s(u)d_{L+r+1,L+k+1}\right].$$

Then

$$\frac{[A_{1k}]^T}{|D|}\left[\frac{1}{\theta}\int_0^\theta b(t,u)dt\right]=\frac{1}{|D|}\left[B_0(u)\sum_{k=1}^{2L+1}d_{k1}A_{k1}+\sum_{r=1}^{L}\left[B_r^c(u)\sum_{k=1}^{2L+1}d_{k,r+1}A_{k1}+\right.\right.$$

$$\left.\left.+B_2^s(u)\sum_{k=1}^{2L+1}d_{k,r+L+1}A_{k1}+\right]\right]=B_0(u), \qquad (12)$$

$$\frac{[A_{p+1,k}]^T}{|D|}\left[\frac{1}{\theta}\int_0^\theta b(t,u)dt\begin{Bmatrix}\cos\omega_r t\\ \sin\omega_r t\end{Bmatrix}dt\right]=\frac{1}{|D|}\left[B_0(u)\sum_{k=1}^{2L+1}d_{k1}\begin{Bmatrix}A_{k,p+1}\\ A_{k,L+p+1}\end{Bmatrix}+\right.$$

$$\left.+\sum_{r=1}^{L}\left[B_r^c(u)\sum_{k=1}^{2L+1}d_{k,r+1}\begin{Bmatrix}A_{k,p+1}\\ A_{k,L+p+1}\end{Bmatrix}+B_r^s(u)\sum_{k=1}^{2L+1}d_{k,r+1}\begin{Bmatrix}A_{k,p+1}\\ A_{k,L+p+1}\end{Bmatrix}\right]\right]=\begin{Bmatrix}B_p^c(u)\\ B_p^s(u)\end{Bmatrix}. \qquad (13)$$

It was taken into account that

$$\sum_{r=1}^{2L+1}d_{rk}A_{rj}=\begin{cases}|D|, & k=j,\\ 0, & k\neq j.\end{cases}$$

Taking into consideration ratios (9)–(13) for biases $\varepsilon\left[\hat{b}(t,u)\right]=E\hat{b}(t,u)-b(t,u)$ LSM-estimators of covariance function are

$$\hat{b}(t,u)=\hat{B}_0(u)+\sum_{l=1}^{L}\left(\hat{B}_l^c(u)\cos\omega_l t+\hat{B}_l^s(u)\sin\omega_l t\right).$$

Find:

$$\varepsilon\left[\hat{b}(t,u)\right]=\varepsilon\left[\hat{B}_0(u)\right]+\sum_{l=0}^{L}\left[\varepsilon\left[\hat{B}_l^c(u)\right]\cos\omega_l t+\varepsilon\left[\hat{B}_l^s(u)\sin\omega_l t\right]\right],$$

where

$$\varepsilon\left[\hat{B}_0(u)\right]=\frac{1}{|D|}\sum_{k=0}^{2L}\varepsilon\left[\tilde{B}_k(u)\right]A_{k+1,1},$$

$$\varepsilon\left[\hat{B}_l^c(u)\right]=\frac{1}{|D|}\sum_{k=0}^{2L}\varepsilon\left[\tilde{B}_k(u)\right]A_{k+1,l+1},$$

$$\varepsilon\left[\hat{B}_l^s(u)\right]=-\frac{1}{|D|}\sum_{k=0}^{2L}\varepsilon\left[\tilde{B}_k(u)\right]A_{k+1,l+L+1},$$

herewith

$$\varepsilon\left[\tilde{B}_0(u)\right]=\frac{1}{\theta}\int_0^\theta\varepsilon_\zeta(t,u)dt,$$

$$\varepsilon\left[\tilde{B}_k(u)\right]=\frac{1}{\theta}\int_0^\theta\varepsilon_\zeta(t,u)\cos\omega_k tdt,$$

$$\varepsilon\left[\tilde{B}_{k+L}(u)\right]=\frac{1}{\theta}\int_0^\theta\varepsilon_\zeta(t,u)\sin\omega_k tdt.$$

Quantity for zero correlation component estimator increases in the first part of function (10) and is the following

$$\varepsilon_1\left[\tilde{B}_0(u)\right]=\frac{1}{|M|}\Bigg[\frac{M_{11}}{\theta^2}\int_0^\theta\int_0^\theta b(t,s-t)dtds+\sum_{q=1}^{\tilde{L}}\frac{M_{11}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\cos\omega_q tdtds+$$

$$+\frac{M_{1,q+\tilde{L}+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\sin\omega_q tdtds+\sum_{p=1}^{\tilde{L}}\Bigg[\frac{M_{p+1,1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\cos\omega_p sdtds+$$

$$+\sum_{q=1}^{\tilde{L}}\Bigg[\frac{M_{p+1,q+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\cos\omega_p s\cos\omega_q tdtds+\frac{M_{p+1,q+\tilde{L}+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\times$$

$$\times\cos\omega_p t\sin\omega_q sdtds\Bigg]+\frac{M_{p+\tilde{L}+1,1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\sin\omega_p sdtds+$$

$$+\sum_{q=1}^{\tilde{L}}\Bigg[\frac{M_{p+\tilde{L}+1,q}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\sin\omega_p s\cos\omega_q tdtds+$$

$$+\frac{M_{p+\tilde{L}+1,q+\tilde{L}+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\sin\omega_p s\sin\omega_q tdtds\Bigg]\Bigg]\Bigg].\qquad(14)$$

Analogic quantities for cosine and sine covariance components are determined by the formulae:

$$\begin{Bmatrix} \varepsilon_1\left[\tilde{B}_k^c(u)\right] \\ \varepsilon_2\left[\tilde{B}_k^s(u)\right] \end{Bmatrix} = \frac{1}{|M|}\left[\frac{M_{11}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix} dtds + \sum_{q=1}^{\tilde{L}}\left[\frac{M_{1,q+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\times\right.\right.$$

$$\left.\times\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix}\cos\omega_q t dtds + \frac{M_{1,q+\tilde{L}+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix}\sin\omega_q t dtds\right] +$$

$$+\sum_{q=1}^{\tilde{L}}\left[\frac{M_{p+1,q+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix}\cos\omega_q t\cos\omega_p s dtds +\right.$$

$$\left.+\frac{M_{p+1,q+\tilde{L}+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix}\sin\omega_q t\cos\omega_p s dtds\right] +$$

$$+\frac{M_{p+\tilde{L}+1,1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix}\sin\omega_p s dsdt + \sum_{q=1}^{\tilde{L}}\left[\frac{M_{p+\tilde{L}+1,q+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\times\right.$$

$$\times\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix}\cos\omega_q t\cos\omega_p s dtds + \frac{M_{p+\tilde{L}+1,q+\tilde{L}+1}}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix}\times$$

$$\left.\left.\times\sin\omega_q t\sin\omega_p s dtds\right]\right]. \quad (15)$$

As it seen from the obtained ratios each of quantities $\varepsilon_1[\tilde{B}_0(u)]$ and $\varepsilon_1[\tilde{B}_k^{c,s}(u)]$ depend on integrals of type

$$\frac{1}{\theta^2}\int_0^\theta\int_0^\theta b(s,t-s+u)\begin{Bmatrix}\cos\omega_k t & \cos\omega_p s \\ \sin\omega_k t & \sin\omega_p s\end{Bmatrix} dtds =$$

$$= \frac{1}{\theta^2}\sum_{r=-L}^{L}\int_0^\theta e^{i\omega_r s}\left[\int_{-s}^{\theta-s} B_r(u_1+u)\begin{Bmatrix}\cos\omega_k t & \cos\omega_p s \\ \sin\omega_k t & \sin\omega_p s\end{Bmatrix} du_1\right] ds .$$

Similar integrals are obtained also by the analysis of quantities

$$\varepsilon_2\left[\tilde{B}_0(u)\right] = \frac{1}{\theta}\int_0^\theta E\left[\mathring{\hat{m}}(t)\mathring{\hat{m}}(t+u) - \mathring{\hat{m}}(t+u)\mathring{\xi}(t)\right] dt , \quad (16)$$

$$\begin{Bmatrix} \varepsilon_2\left[\tilde{B}_k^c(u)\right] \\ \varepsilon_2\left[\tilde{B}_k^s(u)\right] \end{Bmatrix} = \frac{1}{\theta}\int_0^\theta E\left[\mathring{\hat{m}}(t)\mathring{\hat{m}}(t+u) - \mathring{\hat{m}}(t+u)\mathring{\xi}(t)\right]\begin{Bmatrix}\cos\omega_k t \\ \sin\omega_k t\end{Bmatrix} dt . \quad (17)$$

Those quantities (14)–(17) tend to zero if condition (18) is satisfied

$$\lim_{|u|\to\infty} B_k(u) = 0 \;\; \forall k \in Z . \quad (18)$$

Therefore LSM-estimator of covariance function is asymptotically unbiased and estimator bias when θ is finite, is caused not by the leakage effect, by the previous mean function estimation.

To calculate estimator variance

$$\hat{b}(t,u)=\frac{1}{|D|}\left[\sum_{l=0}^{2L}\tilde{B}_l(u)A_{l+1,1}+\sum_{r=1}^{L}\left[\sum_{l=0}^{2L}\left[\tilde{B}_l(u)A_{l+1,r+1}\cos\omega_r t+\tilde{B}_l(u)A_{l+1,1+r+1}\sin\omega_r t\right]\right]\right]$$

rewrite it in the form

$$\hat{b}(t,u)=\frac{1}{|D|}\sum_{l=0}^{2L}\tilde{B}_l(u)f_l(t),$$

where

$$f_l(t)=A_{l+1,1}+\sum_{r=1}^{L}\left[A_{l+1,\mathrm{r}+1}\cos\omega_r t+A_{l+1,L+\mathrm{r}+1}\sin\omega_r t\right].$$

Then

$$D\left[\hat{b}(t,u)\right]=E\left[\hat{b}(t,u)-E\hat{b}(t,u)\right]^2=\frac{1}{|D^2|}\sum_{k,l=0}^{2L}R_{\tilde{B}_k\tilde{B}_r}(u)f_l(t)f_k(t),$$

where $R_{\tilde{B}_k\tilde{B}_r}(u)$ is the covariance of quantities $\tilde{B}_k(u)$ and $\tilde{B}_l(u)$: $R_{\tilde{B}_k\tilde{B}_r}(u)=$ $=E\tilde{B}_k(u)\tilde{B}_l(u)-E\tilde{B}_k(u)\tilde{B}_l(u)$. For Gauss processes in the first approximation:

$$R_{\tilde{B}_k\tilde{B}_l}(u)=\frac{1}{\theta^2}\int_0^\theta\int_0^\theta E\left[\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t+u)\overset{\circ}{\xi}(s)\overset{\circ}{\xi}(s+u)-b(t,u)b(s,u)\right]\times$$

$$\times\begin{Bmatrix}\cos\omega_k t & \cos\omega_l s\\ \sin\omega_k t & \sin\omega_l s\end{Bmatrix}dtds=\frac{1}{\theta^2}\int_0^\theta\int_0^\theta b_\eta(s,s-t,u)\begin{Bmatrix}\cos\omega_k t & \cos\omega_p s\\ \sin\omega_k t & \sin\omega_p s\end{Bmatrix}dtds.$$

Quantity $b_\eta(s,t-s+u)$ is the covariance function of process $\overset{\circ}{\eta}(t,u)=\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t+u)$:

$$b_\eta(s,s-t,u)=b(t,s-t)b(t+u,s-t)+b(t,s-t+u)b(t+u,s-t-u).$$

Let us introduce new integration variable $u_1=s-t$ and change its order:

$$\frac{1}{\theta^2}\int_0^\theta\int_0^\theta b_\eta(s,s-t,u)\begin{Bmatrix}\cos\omega_k t & \cos\omega_p s\\ \sin\omega_k t & \sin\omega_p s\end{Bmatrix}dtds=$$

$$=\frac{1}{\theta^2}\int_0^\theta\int_{-t}^{\theta-t} b_\eta(t,u_1,u)\begin{Bmatrix}\cos\omega_k t & \cos\omega_l(t+u_1)\\ \sin\omega_k t & \sin\omega_l(t+u_1)\end{Bmatrix}du_1dt=$$

$$=\frac{1}{\theta^2}\int_{-\theta}^{0}\int_{-u_1}^{\theta} b_\eta(t,u_1,u)\begin{Bmatrix}\cos\omega_k t & \cos\omega_l(t+u_1)\\ \sin\omega_k t & \sin\omega_l(t+u_1)\end{Bmatrix}dtdu_1+$$

$$+\frac{1}{\theta^2}\int_0^\theta\int_0^{\theta-u_1} b_\eta(t,u_1,u)\begin{Bmatrix}\cos\omega_k t & \cos\omega_l(t+u_1)\\ \sin\omega_k t & \sin\omega_l(t+u_1)\end{Bmatrix}dtdu_1.$$

Since covariance function $b_\eta(t,u_1,u)$ is determined by the product of biperiodical functions $b(t,u)$, it is also biperiodical function and for infinite series (2) can be represented in the form

$$b_\eta(t,u_1,u) = \sum_{k,l\in Z} \tilde{B}_{kl}(u_1,u)\, e^{i\lambda_{kl}t} . \quad (19)$$

It is easy to find quantity $\tilde{B}_{kl}(u_1,u)$, on the basis of series (2). We obtain:

$$b(t,u_1)b(t+u,u_1) = \sum_{k_1,l_1\in Z}\sum_{k_2,l_2\in Z} B_{k_1,l_1}(u_1)\overline{B_{k_2,l_2}(u_1)}\, e^{i\lambda_{k_1-k_2,l_1-l_2}t} e^{-i\lambda_{k_2l_2}u} =$$

$$= \sum_{r_1,r_2\in Z} e^{i\lambda_{r_1r_2}t} \sum_{k_1,l_1\in Z} B_{k_1+r_1,l_1+r_2}(u_1)\overline{B_{k_1,l_1}(u_1)}\, e^{-i\lambda_{k_1+l_1}u} , \quad (20)$$

$$b(t,u_1+u)b(t+u,u_1-u) = \sum_{r_1,r_2\in Z} e^{i\lambda_{r_1r_2}t} \sum_{k_1,l_1\in Z} B_{k_1+r_1,l_1+r_2}(u_1+u)\overline{B_{k_1l_1}(u)}\, e^{-i\lambda_{k_1l_1}u} , \quad (21)$$

where "$\overline{\ }$" is the conjugation sign. So

$$\tilde{B}_{kl}(u_1,u) = \sum_{k_1,l_1} e^{-i\lambda_{k_1l_1}u}\left[B_{k_1+k,l_1+l}(u_1)\overline{B}_{k_1,l_1}(u_1) + B_{k_1+k,l_1+l}(u_1+u)\overline{B}_{k_1,l_1}(u_1-u)\right]. \quad (22)$$

If Fourier series (2) are finite for covariance function then series (19)–(22) are also finite.

It follows from (22) that limit ratio is true if conditions (18) are fulfilled

$$\lim_{|u|\to\infty} \hat{B}_k(u_1,u) = 0 \ \ \forall k,\ l\in Z .$$

And this means, that $D\left[\hat{b}(t,u)\right]\to 0$ when $\theta\to\infty$, namely LSM-estimator of covariance function is consistent.

As it follows the LSM-estimator of covariance function is asymptotically unbiased and consistent, namely has all necessary features allowing us to built algorithms based on it for real data statistical processing. The main advantage of this LSM-method over the component estimator is the absence of leakage effects, which can cause significant errors of covariance characteristics estimation when combination frequencies have close values. Formulae obtained in this paper for statistic characteristics of LSM-estimator give us an opportunity to calculate processing errors for specific signal types and also compare them with the errors of component estimation.

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