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IMPLEMENTATION OF RADON TRANSFORMATION BY ROTATING 1D INTERFEROMETER

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The use of interferometry to observe objects in three-dimensional space requires a corresponding rank of the system of interferometric base vectors. The paper considers one of the ways to solve such a problem using the rotation of a 1D interferometer at an angle to the rotation axis This, with the exception of special cases of parallelism or perpendicularity of the axes, makes it possible to form a consecutive 3D interferometer. Using the rotation of the Earth, such an interferometer performs Radon transform of the angular structure of the spatial image when observing sources far beyond the size of the interferometer base. For this, the elements of the interferometer should be placed at different latitudes. The obtained analytical expressions show that the two-dimensional representation of one-dimensional projections as a function of the rotation angle then takes the form of a sinogram. A rotating 1D interferometer can be used in a number of fields of science and technology, for example, to solve location problems, in ultrasonic defectoscopy, in technical vision systems, in radio astronomy, etc.

Keywords: *1D interferometer, synthesis of 3D interferometer, sinogram, angular intensity distribution.*

РЕАЛІЗАЦІЯ ПЕРЕТВОРЕННЯ РАДОНА ОБЕРТОВИМ 1D ІНТЕРФЕРОМЕТРОМ

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Для спостереження об’єктів у тривимірному просторі з допомогою інтерферометрії в загальному випадку необхідний відповідний ранг системи векторів баз інтерферометричної системи. Розглянуто послідовний синтез 3D інтерферометра на основі обертання 1D інтерферометра, нахиленого під кутом до осі обертання. Інтерферометр складається з нерухомої, розташованої на осі обертання, і рухомої антен. Під час обертання рухома антена рухається по колу, центр якого також лежить на осі обертання. В цьому випадку вектор інтерферометричної бази описує прямий круговий конус і має незмінну довжину. Завдяки обертанню нахиленого інтерферометра, за винятком його паралельності чи перпендикулярності до осі обертання, послідовно утворюється інтерферометрична система. Ранг системи векторів її інтерферометричних баз тоді рівний трьом. Отриманий в загальному вигляді розв’язок не обмежується вузьким полем зору, не вимагає площинності об’єктної сцени, дозволяє одночасно спостерігати багато об’єктів як близьких, так і віддалених з визначенням віддалі до них. Розглянуто можливості застосування обертового 1D інтерферометра в радіоастрономії. Показано, що за використання обертання Землі такий інтерферометр здійснює перетворення Радона кутової структури просторового зображення під час спостереження джерел, віддалі до яких суттєво перевищують розмір бази інтерферометра. Нахилений інтерферометр реалізують за розміщення його елементів на різних широтах. З отриманих аналітичних виразів випливає, що двовимірне подання одновимірних проєкцій залежно від кута повороту приймає вигляд синограми, при цьому різниця в довготах елементів інтерферометра змінює тільки зміщення початку синограми. Тобто за наближення плоскої хвилі наземний двоелементний інтерферометр характеризується двома параметрами – величиною бази і кутом її нахилу до осі обертання Землі. Обертовий 1D інтерферометр можна застосувати у низці областей науки і техніки, наприклад, для задач локації, в ультразвуковій дефектоскопії, в системах технічного зору, в радіоастрономії.

Ключові слова: *1D інтерферометр, синтез 3D інтерферометра, синограма, кутовий розподіл інтенсивності.*

Introduction. The task of constructing the angular distribution of the intensity of radio radiation of objects in the celestial sphere is one of the main tasks of radio astronomy. For these purposes, interferometry and in particular the technology of aperture synthesis, which is based on the application of the van Zittert–Zernike theorem, is widely used [1]. It boils down to the joint processing of data obtained by many two-element (1D) interferometers. Assuming monochromatic, unpolarized radio radiation and a narrow field of view, the measured values of the complex visibility function are the Fourier components of the sky brightness, where samples (u, v) are projections of the baseline of each pair of antennas on a plane perpendicular to the line of sight [2]. That is, indirect measurement in Fourier space is implemented. At the same time, a local object in the spatial domain requires an infinite number of such measurements to correctly solve the inverse problem. Therefore, various methods of interpolation and processing of incomplete data are used. The quality of the reconstruction results significantly depends on the set of bases of the antenna system, the applied methods, and the satisfaction of the conditions of a number of approximations, including: approximation of a plane wave, approximation of the delta-correlation of the source points, approximation of the object field of view to the plane and limitation of its dimensions, approximation of quasi-monochromaticity, approximation of the homogeneity of the medium of wave propagation, approximation of the completeness of filling the area of interferometric bases. A number of methods have been developed to reduce the impact of not fulfilling these approximations, including undersampling and deconvolution, isoplanatism [3], spectral behavior [4], non-coplanarity [5, 6], direction-dependent calibration effects [7], etc. In addition, the reconstruction of radio images belongs to the class of inverse incorrect problems and ensuring its stability requires special methods [8].

A fundamentally different point of view of the problem is considered in [9], which provides a geometric justification for the method of direct image reconstruction. The resulting solution is not limited to a narrow field of view, does not require the flat scene of the object, and allows simultaneous observation of many objects, both close and distant, with the determination of the distance to them. It is shown that all interferometric systems are reduced to a single canonical form corresponding to the rank of the base vector system. The conclusion that the necessary rank of the interferometric system should be three for use in three-dimensional space is substantiated. For this you need at least four antennas located accordingly. The paper notes the possibility of synthesizing several separate interferometers into one system with the possibility of increasing the rank. This approach was developed in [10], where it is shown that with the help of a 1D interferometer placed on the east-west line and the use of the Earth's rotation, it is possible to observe radio sources whose distance is significantly greater than the size of the interferometric base. In this case, a 2D interferometer is synthesized, and in the approximation of large distances, the 3D space can be projected onto the equatorial plane with a reduced dimension, that is, to 2D. It is shown that with such an observation scheme, the interferometer performs a two-dimensional Radon transformation of the angular distribution of radio radiation intensity. But the very projection of the celestial sphere onto the plane leads to ambiguity – the resulting solutions will be symmetrical relative to this plane, which makes it difficult to observe sources near the celestial equator. In addition, it is not always possible to ensure the placement of the interferometer antennas strictly on the east-west line. Therefore, applying a similar technique – using the rotation of the Earth, we will consider the possibilities of observations with the help of a 1D interferometer at different orientations of the base vector.

Rotating two-element (1D) interferometer. Let the interferometer be formed by two antennas, the first of which (A) is placed at the coordinate origin, and the second (A_1) sets the radius vector of the interferometric base $\vec{d} = \overrightarrow{AA_1}$. By $\vec{r}_1$ we denote the vector from the location point of the antenna A_1 to the point in space specified by the radius vector $\vec{r}$ (Fig. 1). Then:

$$\vec{r} = \vec{d} + \vec{r}_1 . \tag{1}$$

The difference in distances from a point in space to the antennas of the interferometer l is written in the form

$$l = r - r_1 \tag{2}$$

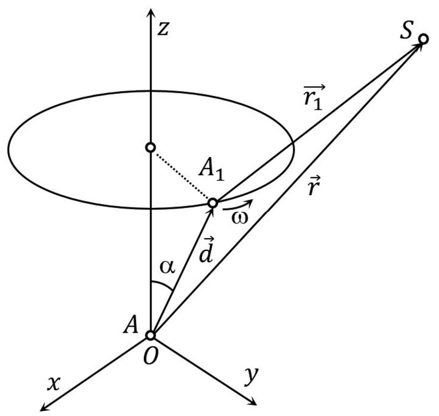

Fig. 1. Rotating 1D interferometer.

(here and in the future, we will denote the length of the corresponding vector by symbols without a vector icon). Note that $-d \le l \le d$.

Let the function $f(\vec{r})$ be defined in the entire space and decay quickly enough to infinity (so that the corresponding improper integrals are convergent). Then the transformation carried out by the interferometer can be conveniently represented using the Dirac delta function

$$G(l, \vec{d}) = \int \delta(r - r_1 - l) f(\vec{r}) d\vec{r} . \tag{3}$$

The integration here is carried out over the entire space, $d\vec{r} = dxdydz$.

As shown in [9], in this form, the two-element interferometer is suitable for solving the problems of object localization in 1D space, but in higher-dimensional spaces the solution of the equation is multi-valued. This is a hyperbola in the case of 2D space and a two-cavity hyperboloid in 3D space, along which the integration in (3) takes place.

Alternatively, the synthesis of a higher-rank interferometer (that is, an interferometer whose base vector system has a higher rank) based on several 1D interferometers is considered there. Provided that the analyzed spatial scene is quasi-stationary, one interferometer can be used as several interferometers, but with a moving antenna A_1, which allows us to consecutively obtain information for interferometer base vectors of different orientations.

In the general case, the law of motion of the base vector of the interferometer can be quite complex to ensure certain qualities of the system, but we will consider a simple rotating interferometer. Let the antenna A_1 perform a uniform motion in a circle parallel to the xOy plane with an angular frequency ω such that changes in the 3D scene can be neglected in one revolution. We place the center of the circle on the Oz axis, and denote the angle between the axis of rotation and the base vector $\vec{d}$ by α . The base vector $\vec{d}$ in this case describes a straight circular cone and has a constant length. Then you can write in the coordinate form

$$\vec{d}(\omega t) = (d \sin\alpha \cos\omega t, d \sin\alpha \sin\omega t, d \cos\alpha) . \tag{4}$$

The rank of the interferometer synthesized in this way can take several values depending on the angle α . So, at $\alpha = 0°$ and at $\alpha = 180°$, the 1D interferometer simply rotates around its axis, which does not affect its response. At $\alpha = 90°$, a 2D interferometer is implemented, this case corresponds to the one discussed in detail in

[10]. At all other values of the angle α, a 3D interferometer is synthesized, which allows us to localize objects in three-dimensional space [9].

Assume that $f(\vec{r})$ is equal to zero everywhere, with the exception of one point S, whose radius vector is in spherical coordinates $\vec{r}_s = (r_s, \theta_s, \phi_s)$, or in rectangular ones:

$$\vec{r}_s = (r_s \sin\theta_s \cos\phi_s, r_s \sin\theta_s \sin\phi_s, r_s \cos\theta_s). \tag{5}$$

Taking into account (1) and (2), the expression for the difference in distances l_s from the point $\vec{r}_s$ to the antennas of the interferometer will have the form

$$l_s(\omega t) = r_s - \sqrt{r_s^2 + d^2 - 2\vec{r}_s \vec{d}(\omega t)}, \tag{6}$$

or, after writing the scalar product $\vec{r}_s \vec{d}(\omega t)$ in coordinate form

$$l_s(\omega t) = r_s - \sqrt{r_s^2 + d^2 - 2r_s d \sin\alpha \sin\theta_s \cos(\omega t - \phi_s) - 2r_s d \cos\alpha \cos\theta_s}. \tag{7}$$

Expression (7) obtained in its general form can be useful in a number of fields of science and technology, for example, for solving location problems, in ultrasonic flaw detection, in systems of technical vision. But to restore radio images using the Earth's rotation and a ground-based interferometer, it is necessary to take into account its complex motion.

Ground-based 1D interferometer using the Earth's rotation. The ground-based interferometer is significantly distant from the Earth's axis of rotation. In addition, together with the Earth, it moves in an orbit around the Sun, performing a complex spiral movement. A detailed account of such movement is possible, but quite difficult, therefore, in radio astronomy, in view of the large distances to radiation sources, the plane wave approximation is widely used [1]. With this approximation, it is assumed that the response of the interferometer does not depend on the position of the interferometer, but only on its orientation relative to the direction of the source. This makes it possible to neglect the movement of the interferometer in space during the observation time and align all the vectors of the interferometric bases $\vec{d}(\omega t)$ at one point. Then the task is reduced to the one discussed above. It is convenient to take the center of the Earth or the pole as the origin of the coordinates (where the antenna A of the interferometer is located). The axis of rotation of the base vector of the interferometer corresponds to the axis of rotation of the Earth, and the angle of inclination of the base vector relative to the axis of rotation is determined based on the ground coordinates of the antennas, taking into account their geographical position. Moreover, if this angle is preserved, then the orientation of the base vector on the Earth's surface is of no special importance. That is, when applying the plane wave approximation, the terrestrial 1D interferometer is characterized by two parameters – the value of the interferometric base and the angle of its inclination to the axis of the Earth rotation.

The plane wave approximation has an important consequence. In this case, the approximation of large distances $r >> d$ is performed for any possible ground bases. This allows us to significantly simplify calculations. Let's rewrite expression (7) as follows:

$$\sqrt{r_s^2 + d^2 - 2r_s d \sin\alpha \sin\theta_s \cos(\omega t - \phi_s) - 2r_s d \cos\alpha \cos\theta_s} = r_s - l_s(\omega t). \tag{8}$$

After squaring both parts of equation (8), we get

$$d^2 - 2r_s d \sin\alpha \sin\theta_s \cos(\omega t - \phi_s) - 2r_s d \cos\alpha \cos\theta_s = l_s^2(\omega t) - 2r_s l_s(\omega t). \tag{9}$$

Let's divide equation (9) by r_s and neglect the expressions $\frac{d^2}{r_s}$ and $\frac{l_s^2(\omega t)}{r_s}$ due to their smallness. Then it will look like:

$$l_s(\omega t) = d \sin\alpha \sin\theta_s \cos(\omega t - \phi_s) + d \cos\alpha \cos\theta_s , \qquad (10)$$

and after dividing both parts by d and marking $\eta_s = \frac{l_s}{d}$ and $\psi = \omega t$:

$$\eta_s(\psi) = \sin\alpha \sin\theta_s \cos(\psi - \phi_s) + \cos\alpha \cos\theta_s . \qquad (11)$$

Thus, taking α as a parameter, the dependences $\eta_s(\psi)$ for sources with coordinates θ_s and ϕ_s are sinusoids with amplitudes proportional to $\sin\theta_s$. In this way, they are similar to the Radon signogram, but have one significant difference. These sinusoids are shifted from zero by an amount proportional to $\cos\theta_s$. If for the 2D interferometer on the sinogram the sources with coordinates (θ_s, ϕ_s) and $(\pi - \theta_s, \phi_s)$ merged into one sinusoid, i.e. there was ambiguity, now they are separated due to the presence of the second term in (11). This is the coordinate part of the transformation. That is, for the source S, the transformation G is different from zero on the line given by the Dirac delta function, the argument of which is determined by expression (11).

Theorem about Radon transform. Let's formulate and prove the following theorem.

Theorem. In the long-distance approximation, a 3D interferometer synthesized by a rotating 1D interferometer realizes the Radon transform of the angular distribution of radiation intensity.

Proof. As it is easy to see, the resulting expression (11) is the scalar product of vectors $\vec{d}$ (4) and $\vec{r}_s$ (5) divided by the product of their lengths. Denoting the unit vector in the direction of the base vector $\vec{n} = \frac{\vec{d}}{d}$ and the unit vector in the direction of the radius vector of the space point $\vec{p} = \frac{\vec{r}}{r}$, on the basis of (11) we write the relative difference in distances from the space point to the elements of the interferometer η in the form

$$\eta = \vec{n}\vec{p} . \qquad (12)$$

Then by analogy with (3)

$$G(\eta, \vec{n}) = \int \delta(\vec{n}\vec{p} - \eta) f(\vec{p}) d\vec{p} , \qquad (13)$$

where $d\vec{p} = dxdydz$. Integration here takes place over a sphere of unit radius, on which the radiation intensity function in angular coordinates $f(\vec{p}) = f(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is defined. To proceed to integration over the entire space, we assume that it is zero everywhere outside the sphere. Then the transformation will take a form that coincides with the definition of the multidimensional Radon transform [11]:

$$G(\eta, \vec{n}) = \int \delta(\vec{n}\vec{r} - \eta) f(\vec{r}) d\vec{r} . \qquad (14)$$

Thus, the transformation G is a Radon transform, which had to be proved.

CONCLUSION

Observations in three-dimensional space using an interferometer generally require a corresponding rank of the base vector system of the interferometric system. One of the ways to solve such a problem is considered using the rotation of a 1D interferometer, which forms an angle with the axis of rotation. The rank of the base vector system of the interferometric system synthesized in this way can be one when the interferometer is placed parallel to the axis of rotation, two when they are perpendicular, and three in other cases. Then the sequential synthesis of a 3D interferometer is possible. For it, an analytical expression of the performed transformation in general form is obtained. Note that the ambiguity inherent in 2D interferometers is eliminated by tilting the interferometer at an angle to the axis of rotation. The two-dimensional representation of one-dimensional projections, depending on the rotation angle, then takes the form of a sinogram.

The use of a rotating 1D interferometer can be useful in a number of fields of science and technology, for example, for solving location problems, in ultrasonic flaw detection, in technical vision systems. But to restore radio images using the Earth's rotation and a ground-based interferometer, it is necessary to take into account its resulting complex motion. First of all, it is the fact that the center of rotation of the interferometer does not lie on the axis of rotation of the Earth. The problem is solved by the application of the plane wave approximation, which is often used in radio astronomy. Then the equations can be significantly simplified. It is shown that in this case the synthesized 3D interferometer performs the Radon transform of the angular structure of the spatial image. This gives the possibility of solving the inverse problem in known and well-researched ways.

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